

Skyrmions with massive pions

Richard A. Battye¹ and Paul M. Sutcliffe²

¹ *Jodrell Bank Observatory, University of Manchester, Macclesfield, Cheshire SK11 9DL, U.K.*

² *Institute of Mathematics, University of Kent at Canterbury, Canterbury CT2 7NF, U.K.*

In the Skyrme model with massless pions, the minimal energy multi-Skyrmions are shell-like, with the baryon density localized on the edges of a polyhedron that is approximately spherical and generically of the fullerene-type. In this paper we show that in the Skyrme model with massive pions these configurations are unstable for sufficiently large baryon number. Using numerical simulations of the full nonlinear field theory, we show that these structures collapse to form qualitatively different stable Skyrminion solutions. These new Skyrmions have a flat structure and display a clustering phenomenon into lower charge components, particularly components of baryon numbers three and four. These new qualitative features of Skyrmions with massive pions are encouraging in comparison with the expectations based on real nuclei.

PACS numbers: 12.39.Dc, 11.10.Lm

The Skyrme model [15] is a nonlinear theory of pions which is an approximate, low energy effective theory of quantum chromodynamics, obtained in the limit of a large number of quark colours [16]. Skyrmions are topological soliton solutions of the model and are candidates for an effective description of nuclei, with an identification between soliton and baryon numbers (for a review see [14]).

Recent investigations [7] have suggested that there will be important qualitative differences between Skyrmions with massive and massless pions. Minimal energy multi-Skyrmions with massless pions are shell-like, with the baryon density localized on the edges of a polyhedron that is approximately spherical and generically of the fullerene-type. However, for massive pions such solutions fail to be bound states for particular baryon numbers, with a strong dependence upon the value of the pion mass [7]. It was suggested that there must be qualitative changes as the pion mass is increased, but the form of the new Skyrminion solutions was not computed. In this paper we address this issue by performing numerical simulations of the full nonlinear field theory. Starting with the massless pion Skyrmions which we perturb in the massive theory, we find that the fullerene structures collapse to form qualitatively different stable Skyrmions. The new Skyrmions have a flat structure and display a clustering phenomenon into lower charge components, particularly components of baryon numbers three and four. These new qualitative features are encouraging in comparison with the expectations based on real nuclei, and provide motivation for a number of further investigations.

The field U of the Skyrme model [15] is an $SU(2)$ -valued scalar with an associated current $R_i = (\partial_i U)U^\dagger$. In this paper we are only concerned with static fields, so we can define the Skyrme model by its energy

$$E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}([R_i, R_j][R_i, R_j]) + m^2 \text{Tr}(1 - U) \right\} d^3x, \quad (1)$$

which is normalized so that the Faddeev-Bogomolny bound reads $E \geq |B|$, where B is the baryon number (or topological charge) given by the degree of the mapping. In the above the energy and length units (which must be fixed by comparison to real data) have been scaled away, leaving only the pion mass parameter m . This parameter is proportional to the (tree-level) pion mass, in scaled units.

The most detailed studies of multi-Skyrmions [6] have assumed massless pions ($m = 0$), and only for very low baryon numbers have massive pions been included [3, 8, 12]. Furthermore, when a non-zero pion mass has been introduced, it has always been set to the experimentally measured value determined by matching to the linearized pion theory. This is the approach adopted in early studies of a single Skyrminion [1] based on reproducing the masses of the proton and delta resonance [2], and leads to the value $m = 0.526$. However, this approach has recently been re-examined [4, 9] by removing some of the assumed approximations, with the result that the proton and delta masses can only be reproduced if the pion mass is taken to be larger than roughly twice the experimentally measured value. One interpretation of this development is that the pion mass parameter in the Skyrme model should be regarded as a renormalized pion mass, and therefore not be fixed by the experimentally measured value, but instead treated as a free parameter to be adjusted to best reproduce the properties of nuclei. As we are mainly interested in qualitative phenomena in this paper, and motivated by the results in Ref.[4], we shall set $m = 1$ for most of our study. We shall briefly discuss how our results are modified by alternative values, but an in-depth quantitative analysis of multi-Skyrmion properties as a function of m is beyond the scope of the current investigation.

For massless pions the minimal energy Skyrmions have been obtained for all $B \leq 22$ [6], and are well-approximated by the rational map ansatz [11]. This involves a decomposition of the field into a radial and

angular dependence, with the angular dependence determined by a rational map between Riemann spheres. One of the reasons it is such a good approximation is due to the roughly spherical shell-like distribution of the energy density of these Skyrmions.

The vacuum value $U = 1$ is attained at spatial infinity and the number of points in space (counted with multiplicity) at which the anti-vacuum value $U = -1$ is attained must equal the baryon number B , by simple topological arguments. These anti-vacuum points are, therefore, a useful characterization of the field, and in particular are the locations of a set of single Skyrmions if they are all well-separated. In the rational map approximation all B points are coincident at the centre of the shell, and this appears to be a property shared by the exact solutions for massless pions when there is a large amount of symmetry (axial or Platonic, for example). However, when there is only dihedral symmetry or less, these B anti-vacuum points can split into B distinct points consistent with any dihedral symmetry. For example, the $B = 9$ solution has D_{4d} symmetry and one anti-vacuum point is at the origin with the other eight on the vertices of a regular octagon in the plane orthogonal to the main symmetry axis and containing the origin. The locations of the anti-vacuum points in the case of massive pions will be discussed below.

In order to test the stability of shell-like solutions we have used the same methods described in detail in Ref.[6] to numerically relax field configurations to static solutions which are local energy minima of the Skyrme model with $m = 1$. The results presented here used grids containing 101^3 points with a lattice spacing $dx = 0.1$, though other grid sizes and lattice spacings were tested for comparison.

As an initial condition we take the minimal energy charge B Skyrmion for massless pions (actually we use its rational map approximation, but this is good enough) and perturb it by squashing it by 20% in a direction which is not aligned with any symmetry axis of the Skyrmion. This ensures that the initial configuration has no exact symmetry. The results of energy relaxations with $m = 1$ on a parallel supercomputer are presented below.

For $B \leq 9$ we find that essentially the same configurations as in the massless case are recovered, though the Skyrmions are now smaller in size and are exponentially (rather than algebraically) localized, which are the obvious consequences of massive pions. There is spherical symmetry for $B = 1$, axial symmetry for $B = 2$, Platonic symmetry for $B = 3, 4, 7$, and dihedral symmetry for $B = 5, 6, 8, 9$. In the examples with dihedral symmetry the Skyrmion is slightly squashed in the direction of the main symmetry axis, so that it appears flat in comparison to the massless case. We shall elaborate on this a little later when we consider the $B = 8$ example in more detail. For now we simply remark that the squashed na-

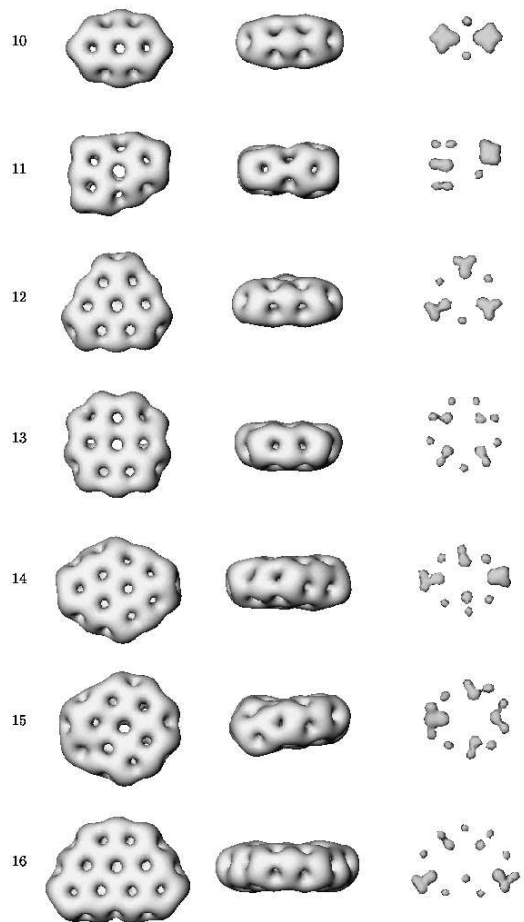


FIG. 1: Skyrmion solutions for $10 \leq B \leq 16$, with pion mass parameter $m = 1$. The first two columns are baryon density isosurface plots from two different viewing angles, and the third column displays isosurfaces where $\frac{1}{2}\text{Tr}(U) = -0.9$. This allows the identification of the anti-vacuum points and their cluster structure. All plots are displayed using the same scale.

ture of the relaxed configuration is not related to the squashing used to perturb the initial condition.

For $B \geq 10$ we find that the spherical shell-like configurations are unstable and they relax to solutions which are much less symmetric and have a remarkably flat structure. We present these solutions, to scale, in Fig. 1 for $10 \leq B \leq 16$. In this figure the first two columns are baryon density isosurface plots from two different viewing angles, and the third column displays isosurfaces where $\frac{1}{2}\text{Tr}(U) = -0.9$. This allows us to identify the anti-vacuum points, together with the nature of their clustering. Choosing an isosurface value closer to -1 allows us to pinpoint the anti-vacuum points more accurately (and in particular count them to confirm that there are B in each case) but the information of how they group into clusters is then lost, and this will be an important feature as we shall see below.

TABLE I: The symmetry group G and the energy per baryon E/B , for Skyrmions with $10 \leq B \leq 16$ and pion mass parameter $m = 1$.

B	10	11	12	13	14	15	16
G	D_{2h}	\mathbb{Z}_2	C_3	\mathbb{Z}_2	\mathbb{Z}_2	C_2	\mathbb{Z}_2
E/B	1.290	1.297	1.289	1.292	1.288	1.291	1.288

The views from the top and the side, displayed in the first two columns of Fig. 1, confirm the flat planar structure of these solutions. The instability of spherical shells for massive pions originates from the fact that the interior of the shell is a large volume over which the Skyrme field is close to the anti-vacuum value $U = -1$, and the pion mass term makes this energetically unfavourable [7]. The new Skyrmion solutions reduce this energy contribution by being very flat and hence reducing the volume to surface area ratio of the configuration.

Clearly, flat planar structures can have at most a dihedral symmetry, so they will generally be less symmetric than the spherical structures, which often have Platonic symmetries. The symmetries of the new Skyrmions are presented in Table I, where \mathbb{Z}_2 denotes only a reflection symmetry. It can be seen from this Table that these Skyrmions typically have very little symmetry, even for planar arrangements.

Also shown in Table I is the energy per baryon E/B for each solution. This demonstrates that the energy per baryon is similar for all baryon numbers in this range, though there is a clear tendency for the energy to be slightly higher for odd baryon numbers. For each of the odd baryon numbers in Table I the energy per baryon is higher than for the even baryon number solutions obtained by either adding or subtracting one from the baryon number. This is a new feature not shared by Skyrmions with massless pions, where often it is solutions with odd baryon numbers that have unusually low energy [6]. This new feature is an encouraging sign in relating to real nuclei, where even mass numbers have larger binding energies and are more abundant than those with odd mass numbers.

It should be noted that we have found it more difficult to accurately compute energies in the massive pion theory than in the massless one, since the pion mass produces a more rapid spatial variation of the fields. Thus some caution should be applied in trusting the third decimal place in the energies presented in Table I.

To interpret the structure of the solutions presented in Fig. 1 it is helpful to consider the clustering of the anti-vacuum points displayed in the third column. For example, for $B = 10$ there are two groups of four anti-vacuum points and two single points. This suggests that the configuration should be interpreted as being composed of two charge four Skyrmions and two single Skyrmions. Recalling that the baryon density of the $B = 4$ Skyrmion

is localized on the edges of a cube, this interpretation is consistent with the baryon density plot in the first column, where two deformed cube-like structures are visible.

As another example, the anti-vacuum points of the $B = 12$ Skyrmion are clustered in three groups of three, in an equilateral triangle, with three single Skyrmions on the dual triangle. This suggests that this $B = 12$ Skyrmion contains three tetrahedra (the minimal $B = 3$ Skyrmion having its baryon density localized on the edges of a tetrahedron) and three single Skyrmions. This is not so obvious from the baryon density isosurface presented in the first column of Fig. 1, however, we were able to confirm this interpretation as follows. A simulation of the full nonlinear time dependent Skyrme equations was performed (using the numerical code described in detail in Ref.[6]) with an initial condition consisting of the new $B = 12$ Skyrmion, but with an overall size rescaling so that it was initially too small. During the time evolution this configuration expands and due to the large kinetic energy it breaks up into a triangular arrangement of three tetrahedra, plus the three single Skyrmions on the dual triangle, in agreement with the prediction based on the anti-vacuum points. Each tetrahedron has a face parallel to the plane of the triangle and a vertex pointing up from this plane. This breaks the up-down symmetry in the plane and explains why the $B = 12$ Skyrmion has only a C_3 symmetry, not a D_3 symmetry. From the baryon density figure in the first column of Fig. 1 it can be seen that there is a hole in the centre of the top of the Skyrmion, but there is no corresponding hole in the centre at the bottom, as can be confirmed by looking through the top hole.

The Skyrmions with other values of B have a similar cluster structure, with groups of three and four anti-vacuum points often occurring; suggesting that substructures of charges three and four are favoured. This is another encouraging development, since it is known that many nuclei may be described as arrangements of alpha particles. This aspect will be explored elsewhere [5].

Given that these Skyrmions appear to be formed from combinations of smaller charge units, it seems likely that there will be many local minima given by different possible partitions and geometrical arrangements of the smaller charge components. Many of the Skyrmions shown in Fig. 1 have very little symmetry and this appears to be due to the fact that there is a partial clustering together with several single anti-vacuum points. Thus it may be that these stable solutions are only local minima and different global minima may exist which have more clustering. It is a computationally expensive task to investigate this issue, since it requires many simulations for each baryon number using a variety of different initial conditions. Such a study is currently underway and the results will be reported in the near future [5].

As discussed in detail in Ref.[7], as the pion mass is increased its effects become more important and hence



FIG. 2: Two views of the baryon density isosurface of the $B = 8$ Skyrmion (a) with $m = 1$; (b) with $m = 2$.

the expectation is that shell-like Skyrmons (including squashed versions) will become unstable at lower baryon numbers. To test this conjecture we consider the $B = 8$ Skyrmon with pion mass parameter $m = 1$ and $m = 2$. In Fig. 2(a) we present two views of the baryon density isosurface of the $B = 8$ Skyrmon with pion mass parameter $m = 1$. It has D_{6d} symmetry and, as mentioned earlier, is essentially the same as in the massless case except for the slight squashing in the direction of the main symmetry axis (which can be seen in the second figure in Fig. 2(a)). In Fig. 2(b) we display two views of the baryon density isosurface for the Skyrmon solution we find when $m = 2$. This is a flat Skyrmon with only a \mathbb{Z}_2 reflection symmetry and is evidently of the same type as the Skyrmons we have found for $B \geq 10$ with $m = 1$. The anti-vacuum points divide into a group of five and another group of three. This confirms our expectation that a larger pion mass yields a qualitative change in the structure and symmetry of Skyrmons at lower baryon numbers.

We have explored a range of different pion masses ($m = \frac{1}{4}, \frac{1}{2}, 2$) in addition to the $m = 1$ results presented in detail. Using the $m = 1$ solutions as initial conditions, we found that the solutions presented in Fig. 1 are essentially reproduced for $m = 2$ and $m = \frac{1}{2}$, but not for $m = \frac{1}{4}$ where the solutions reverted to those found for $m = 0$.

In summary, we have demonstrated that there are important qualitative differences between Skyrmons with massless and massive pions. This has crucial implications for the ultimate long term aim of comparing the Skyrme model of nuclei with real data. The qualitative changes we have found suggest that there is an increased hope of reproducing some of the most important properties of nuclei, and motivates further investigations of the properties of the solutions, some of which are currently underway.

In a zero mode quantization of Skyrmons the symmetry of the classical solution plays an important role in providing constraints on the spin, isospin and parity quantum numbers. For massless pions this has been studied in detail [13] for baryon numbers upto 22, with some success in comparing with experimental data, particularly for low baryon numbers. However, there are a number of discrepancies which arise due to the large symmetry groups of the solutions. The results we have

presented in this paper reveal that Skyrmons with massive pions are much less symmetric, so it would be interesting to calculate the new constraints on the quantum numbers that now arise.

Acknowledgements : Many thanks to Steffen Krusch and Nick Manton for useful discussions. This work was supported by the PPARC special programme grant ‘‘Classical Lattice Field Theory’’. The parallel computations were performed on COSMOS at the National Cosmology Supercomputing Centre in Cambridge. During the preparation of this paper we became aware of independent work in progress [10] which has a substantial overlap with the work reported here. There is a qualitative agreement between the two sets of results and we thank Conor Houghton and Shane Magee for informing us of their work.

-
- [1] G. S. Adkins and C. R. Nappi, The Skyrme model with pion masses, *Nucl. Phys.* **B233**, 109 (1984).
 - [2] G. S. Adkins, C. R. Nappi and E. Witten, Static properties of nucleons in the Skyrme model, *Nucl. Phys.* **B228**, 552 (1983).
 - [3] C. Barnes, W. Baskerville and N. Turok, Normal modes of the $B = 4$ Skyrme soliton, *Phys. Rev. Lett.* **79**, 367 (1997); Normal mode spectrum of the deuteron in the Skyrme model, *Phys. Lett.* **B411**, 180 (1997).
 - [4] R. A. Battye, S. Krusch and P. M. Sutcliffe, Spinning Skyrmons and the Skyrme parameters, *Phys. Lett.* **626**, 120 (2005).
 - [5] R. A. Battye, N. S. Manton and P. M. Sutcliffe, Skyrmons and Nuclei, *in preparation*.
 - [6] R. A. Battye and P. M. Sutcliffe, Symmetric Skyrmons, *Phys. Rev. Lett.* **79**, 363 (1997); Solitonic fullerene structures in light atomic nuclei, *Phys. Rev. Lett.* **86**, 3989 (2001); Skyrmons, fullerenes and rational maps, *Rev. Math. Phys.* **14**, 29 (2002).
 - [7] R. A. Battye and P. M. Sutcliffe, Skyrmons and the pion mass, *Nucl. Phys.* **B705**, 384 (2005).
 - [8] E. Braaten, S. Townsend and L. Carson, Novel structure of static multisoliton solutions in the Skyrme model, *Phys. Lett.* **B235**, 147 (1990).
 - [9] C. J. Houghton and S. Magee, A zero-mode quantization of the Skyrmon, *Phys. Lett.* **B632**, 593 (2006).
 - [10] C. J. Houghton and S. Magee, *in preparation*.
 - [11] C. J. Houghton, N. S. Manton and P. M. Sutcliffe, Rational maps, monopoles and Skyrmons, *Nucl. Phys.* **B510**, 507 (1998).
 - [12] V. Kopeliovich, Characteristic predictions of topological soliton models, *J. Exp. Theor. Phys.* **93**, 435 (2001).
 - [13] S. Krusch, Homotopy of rational maps and the quantization of Skyrmons, *Ann. Phys.* **304**, 103 (2003).
 - [14] N. S. Manton and P. M. Sutcliffe, *Topological Solitons*, Cambridge University Press (2004).
 - [15] T. H. R. Skyrme, A nonlinear field theory, *Proc. R. Soc. Lond.* **A260**, 127 (1961).
 - [16] E. Witten, Global aspects of current algebra, *Nucl. Phys.* **B223**, 422 (1983); Current algebra, baryons, and quark confinement, *ibid* **B223**, 433 (1983).